

Critical exponents for granular phase transitions

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Abstract. – The solid-fluid phase transition of a granular material shaken horizontally is investigated numerically. We find that it is a second-order phase transition and propose two order parameters, namely the averaged kinetic energy and the averaged granular temperature, to determine the fluidization point precisely. It scales with the acceleration of the external vibration. Using this fluidization point as critical point, we discuss the scaling of the kinetic energy and show that the kinetic energy and the granular temperature show two different universal critical point exponents for a wide range of excitation amplitudes.

Granular materials show a fascinating behaviour under a variety of experimental conditions [1]. Most striking are the different regimes found under vertical shaking: for low excitations sound waves are found, for medium excitations heap formations and convection rolls are visible and for high excitations period doubling can occur which can lead to standing and travelling waves [2]. This behaviour is connected to two visible phase transitions where the strength of the excitation is measured in the dimensionless variable $\Gamma := A_0 \omega / g$, with A_0 and ω standing for the amplitude and the angular frequency of the external shaking and g denoting the gravitational constant. The first transition occurs around $\Gamma = 1$ [3, 4] and the second was found to occur around $\Gamma = 1.8$ in a one-layer system [4]. In a quasi two-dimensional box, the transition points were significantly higher [5]. In the intermediate regime, the bed expansion scales as Γ [4], whereas the supercritical scaling is correlated with $A_0 \omega$ [2, 4, 6, 7]. In this letter, we will focus on a slightly different setup and study the first phase transition from a solid-like to a fluid-like regime of a granular material undergoing *horizontal* vibrations. Our aim will be to quantify this transition point and to show the correct scaling with A_0 and ω . Furthermore, we demonstrate that the averaged kinetic energy and the granular temperature show beautiful scaling over more than one decade with different critical exponents.

The system in mind was first described by Straßburger et al. [8] and an enhanced setup was used in [9] for quantitative measurements which we will sketch briefly. A few layers of commercially available glass ballotonies were filled into a 100 mm long and 0.6 mm wide channel and exposed to horizontal vibrations of the form $A(t) = A_0 \sin(\omega t)$ where $\omega = 2\pi f$. In order to get better defined surface properties for particle-wall contacts a monolayer of smaller spheres was glued onto the channel bottom. Since the channel width is equal to the maximum

particle diameter, one gets a quasi two-dimensional system which can easily be observed from the side. For low excitations, either low frequencies f or small amplitudes A_0 , no relative motion of particles with respect to the container is visible and the spheres form a nearly perfect triangular lattice, sketched in fig. 1a. For higher excitations, relative motion sets in and the lattice breaks up, sketched in fig. 1b. Using this system, the phase diagram for the solid-fluid transition was investigated and it was found that the transition point scales with Γ using experiments and numerical simulations [9]. A summary of the phase diagram is shown in fig. 2, demonstrating that the numerical model accurately describes the experiment and the slight shift could be attributed to the higher wall friction present in the experiment. For the experimental values, the onset of grain motion is denoted by f_0 and the extrapolated critical frequency by f_c .

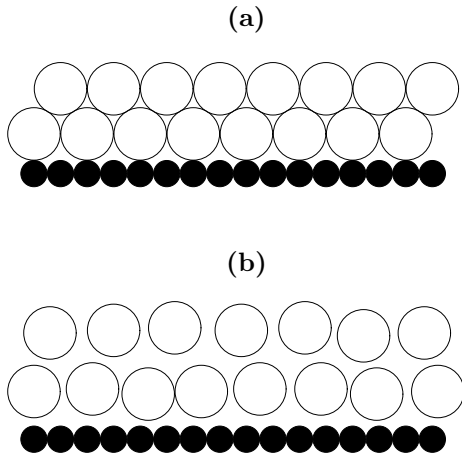


Fig. 1

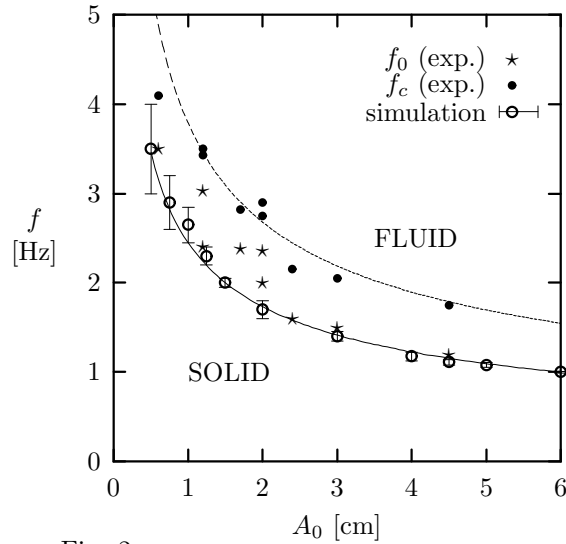


Fig. 2

Fig. 1. – Sketch of the physical system undergoing horizontal vibrations for (a) low and (b) higher excitations.

Fig. 2. – Phase diagram for the solid–fluid transition given in ref. [9]: experimental points are \star (f_0) and \bullet (f_c); data from numerical simulations are \circ with solid line as best fit. Dashed line shows theoretical curve for equal size particles.

We will study the above mentioned system numerically using spherical particles which are bound to move in two dimensions and only interact via contact forces. Periodic boundary conditions are used in the direction of the shaking (horizontal direction) and 342 particles are placed to form two layers initially. A predictor-corrector time integration scheme commonly used in molecular dynamics type simulations is used. The forces acting on particle i , having a radius of $r_i = d_i/2$, during a collision with particle j are in the normal direction (\hat{n})

$$F_{ij}^n = -k_n(r_i + r_j - \vec{r}_{ij}\hat{n}) - \gamma_n \vec{v}_{ij}\hat{n} \quad (1)$$

and in the shear direction (\hat{s})

$$F_{ij}^s = \text{sign}(\vec{v}_{ij}\hat{s}) \cdot \min(\gamma_s \vec{v}_{ij}\hat{s}, \mu |F_{ij}^n|) \quad (2)$$

Here \vec{r}_{ij} stands for the vector joining both centers of mass and \vec{v}_{ij} denotes the relative motion of the two particles. The particles are allowed to rotate since this is well observed experimentally. The parameter k_n is related to the Young modulus and is chosen as 10^6 N/m which is high enough to avoid unphysical results due to the *detachment effect* [10]. The parameter γ_n is related to the experimentally more accessible restitution coefficient e_n and a value of $e_n = 0.75$ was used in all simulations in accordance with the experiment [9]. The relevant parameters entering the shear force are given by the detailed collision experiments conducted by Foerster et al. [11]. Our parameter γ_s was set to such a high value that the shear force was mostly dominated by the Coulomb threshold condition $\mu|F_{ij}^n|$ where a value of $\mu = 0.1$ was used for particle-particle and $\mu_w = 0.13$ for particle-wall contacts. Our force law in the shear direction, eq. (2), is a slight oversimplification with respect to the experimental results in the low shear velocity regime [11] since it does not contain the reversion of rotation. We checked the validity of this approximation by using a shear force that exactly matches the data points given in ref. [11] for glass spheres and even though the transition point shifted slightly, neither the scaling of the transition points with Γ nor the critical point exponents changed their values within the numerical accuracy. A more detailed discussion of the different force laws was given by Schäfer et al. [12]. A system size of 10.08 cm was used and the average particle diameter was $\langle d \rangle = 0.56$ mm with a uniform size variation of ± 0.04 mm. The container bottom consists of smaller particles with an average size of $\langle d \rangle/4$ denoted by black particles in fig. 1.

In order to detect the relative motion between particles, we look at the average kinetic energy of all particles, defined as $e_{\text{kin}}(\tau) := \frac{1}{2N} \sum_{i=1}^N m_i v_i^2(\tau)$, as function of time measured in the moving reference frame of the channel. This signal is averaged over an integer number of external vibration cycles, typically 5-10, to give E_{kin} and we propose that it can be used as an order parameter for the solid-fluid transition in granular assemblies. In fig. 3a, we show how E_{kin} depends on the shaking frequency f measured in Hz, for two different shaking amplitudes A_0 (\bullet – 7 cm, \circ – 3 cm). The errorbars are less than the symbol size. Below a well defined threshold f_c , E_{kin} is approximately zero which indicates that the particles are all at rest and can be viewed as a solid. Above the threshold, a monotonic, drastic increase is visible. Particles are moving around relative to each other and the average density decreases. The state of the system in this regime can be best described by a fluid.

A more accurate method to determine f_c is given by subtracting the bulk motion from the kinetic energy in the following fashion $t_{\text{gran}}(\tau) := \frac{1}{2N} \sum_{i=1}^N m_i (v_i(\tau) - \langle v \rangle)^2$ which is commonly referred to as the granular temperature of the system. Here $\langle v \rangle$ stands for the average particle velocity at time τ . This quantity is also averaged over an integer number of full cycles of the external vibration to give T_{gran} . Below the transition point, this quantity is zero and it is non-zero above it. Since T_{gran} scales linearly with f for a wide frequency range it can easily be extrapolated to the limit $T_{\text{gran}} \rightarrow 0$ to give a very accurate value of f_c . This procedure is demonstrated in fig. 3b for the two shaking amplitudes $A_0 = 7$ cm and $A_0 = 3$ cm and gives values of $f_c = 0.91$ and 1.4 Hz, respectively. When kinetic theories were applied to vibrated beds the granular temperature was found to scale as $T \sim (A_0 \omega)^2$ [5, 7] which is in contrast to our finding and indicates that the current theories do not adequately describe the solid-fluid transition in granular materials.

When the critical frequencies for an amplitude range of 0.5 to 9 cm are plotted on a double logarithmic scale, one finds a scaling of $A_0 f_c^2 = \text{const.}$ which is shown in fig. 4. Such a scaling relation was also found for the transition point for surface instabilities and the onset of convection rolls in a sand pile under *vertical* vibrations [3]. It can be understood by looking at the force balance along the shear direction of one particle on a row of particles. This gives as necessary condition for particle motion $A_0 \omega^2/g > \tan \Theta$ where Θ measures the angle from the vertical of the center of mass of the top particle to the next particle center below it. For

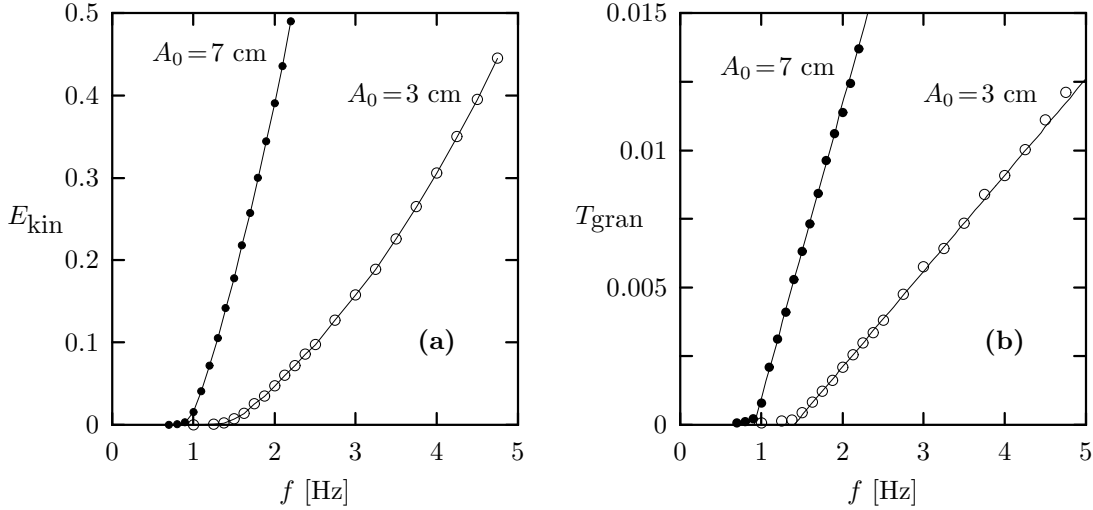


Fig. 3. – Averaged kinetic energy (a) and granular temperature (b) as function of frequency f for two different shaking amplitudes $A_0 = 7$ cm and $A_0 = 3$ cm.

equal size spheres this gives $\Theta_e = 30^\circ$ whereas in our case $\Theta = 11.5^\circ$.

Knowing that the transition point scales as $A_0 f_c^2 = \text{const.}$, we replot in fig. 5 the averaged kinetic energy E_{kin} divided by A_0 as function of the dimensionless quantity $\Gamma' := A_0 f^2 / (g \tan \Theta)$. All data points collapse onto one straight line, giving a transition point of $\Gamma'_c \approx 1.07$. For the vertical shaking experiments, a transition point of $\Gamma'_c \approx 1.2$ was found [3].

The transition point does not depend on the choice of the restitution coefficient in the normal direction, even though it changes the kinetic energy and has a dramatic effect on the absolute value of the granular temperature. On the other hand, changing the Coulomb threshold μ shifts the critical point when a high enough value of γ_s is used. For $\mu = 0$, the transition frequency is lowered by roughly 0.5 Hz; whereas increasing it to $\mu = 1$ increases f_c by up to 1 Hz as well. Changing the system size to 7.56 or 15.12 cm did neither change the transition point nor the absolute values of the averaged kinetic energy and granular temperature.

We will now turn to the main issue of this letter, namely the calculation of critical-point exponents for the solid–fluid phase transition in granular material. The sharp transition at f_c from a zero value kinetic energy (or granular temperature) to a monotonically increasing value with increasing frequency suggests that the solid–fluid phase transition in a granular material undergoing vertical vibrations can be described by a second-order phase transition. This is stressed by the fact that no hysteresis can be found in the numerical simulations when increasing or decreasing the driving frequency, f . It was shown above that the kinetic energy or the granular temperature can both equally well be used as order parameter to characterize this transition. We use

$$\epsilon := \frac{f - f_c}{f_c}$$

as dimensionless variable to describe the behavior near the critical point f_c [13]. The critical exponent λ is then given by

$$\lambda := \lim_{\epsilon \rightarrow 0} \frac{\ln h(\epsilon)}{\ln \epsilon}$$

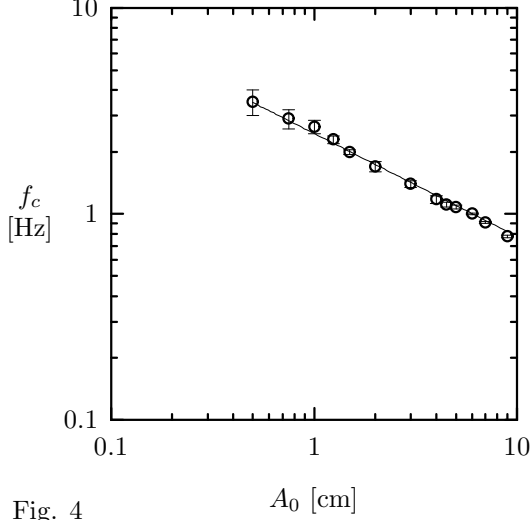


Fig. 4

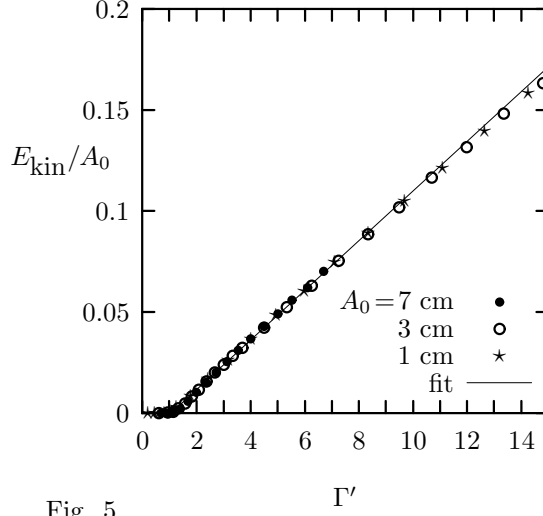


Fig. 5

Fig. 4. – Critical frequency as function of the external shaking amplitude A_0 Fig. 5. – Rescaled averaged kinetic energy as function of dimensionless quantity $\Gamma' := A_0 f^2 / (g \tan \Theta)$.

where $h(\epsilon)$ stands for either the kinetic energy E_{kin} or the granular temperature T_{gran} . Assuming the existence of this limit, we can approximate $h(\epsilon)$ in the vicinity above the critical point by

$$h(\epsilon) \sim \epsilon^\lambda .$$

We applied this technique to 13 different shaking amplitudes A_0 in the range 0.5 to 9.0 cm and found an universal exponent for the scaling of the kinetic energy of 1.25 ± 0.15 and of the granular temperature of 1.0 ± 0.15 . The error bars stem from the possible error in the critical frequency. But no systematic trend of the exponent for decreasing or increasing values of A_0 can be observed. In fig. 6, we show this scaling for shaking amplitudes of $A_0 = 7, 3$ and 1 cm using either the kinetic energy and an exponent of 1.25 (a) or the granular temperature and an exponent of 1.0 (b) for all three curves. For each curve, we used values of ϵ ranging over at least one order of magnitude. For $A_0 = 3$ cm, we extended our data points to very high excitation amplitudes and find a different exponent for values of $\epsilon > 3$ which might indicate that the transition from the fluid to a more gas-like state takes place in this regime [5]. For the lowest shaking amplitude, the left most point shows a slight deviation from the fit which can be attributed to a less accurate value for the transition frequency for this excitation amplitude and finite size effects.

In this letter, we investigated numerically the solid–fluid phase transition of a granular material exposed to horizontal shaking. By looking at the averaged kinetic energy or the granular temperature of all particles, we found that these quantities are only non-zero above a well defined transition frequency which we called critical frequency and showed no hysteresis. This suggests that this phase transition is of second-order and we presented the correct scaling of the transition frequency with respect to the shaking amplitude. In the vicinity of the critical point, we calculated the critical exponents associated with the kinetic energy and the granular temperature and found an universal exponent of 1.25 for the first and of 1.0 for the latter quantity. The same technique can probably be applied to the solid–fluid phase transition

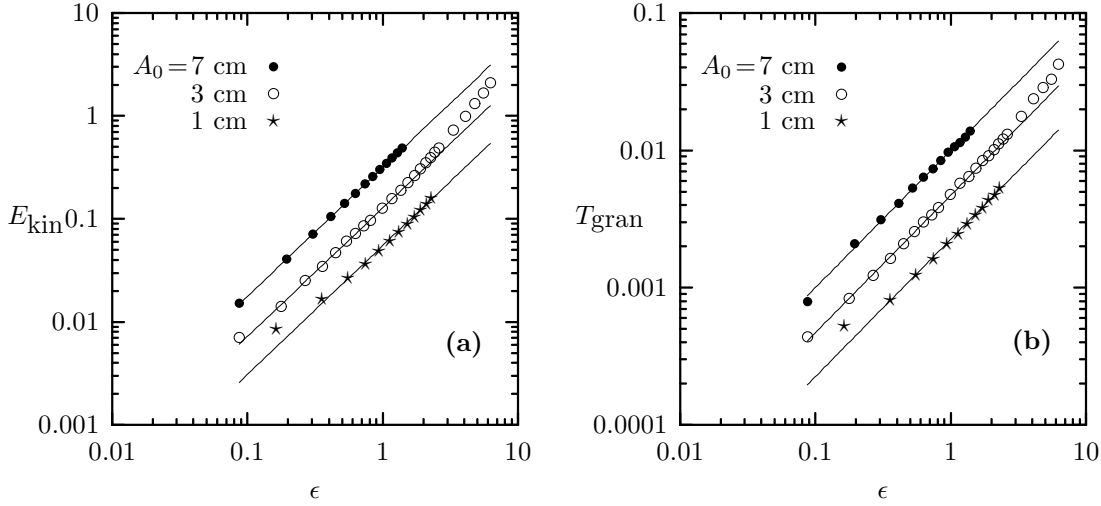


Fig. 6. – Scaling of the kinetic energy (a) and the granular temperature (b) as function of dimensionless parameter $\epsilon := (f - f_c)/f_c$. The universal exponent λ is 1.25 ± 0.15 in case (a) and 1.0 ± 0.15 in case (b).

under *vertical* vibration and it would be interesting to compare the exponents with our results.

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